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SEARCH AND EVASION GAMES

by

Roger G. Schroeder

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Rear Admiral E. J. O'Donnell, USN, Superintendent

Dr. R. F. Rinehart, Academic Dean

ABSTRACT:

We develop some two-person zero-sum game formulations of search and evasion problems. By employing a game theoretic approach, we allow the hider, as well as the searcher, to choose a strategy. This is in contrast to most search models which assume a stationary or passive hider. Both non-sequential and sequential search games are investigated. Some interesting aspects of the non-sequential game and an example of an antisubmarine search problem are given. The sequential games consist of a sequence of moves. When the players move, they not only determine a payoff but also the probability that the game terminates before the next move. When at most a finite number of moves is allowed, we prove that a solution may be found by solving a recursive sequence of matrix games. When the number of moves is not bounded, the game is characterized by a special type of non-linear program. The solution to this program can be approximated by successive perturbations of a related linear program. Finally, we obtain the result that a pair of strategies minimaxes the expected duration of the game if and only if these strategies also maximin the probability of termination in one step.

Prepared by: R. G. Schroeder

Approved by:

J. R. Borsting,

Chairman, Department of Operations Analysis

Released by:

C. E. Menneken,

Dean of Research Administration

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1. INTRODUCTION

We investigate some two-person zero-sum games which typically arise in search and evasion settings. One of the classical problems in the theory of search is to determine an optimal division of search effort among n-cells.[1] We explore this problem when both the searcher (P1) and the hider (P2) can choose a cell on each move. Here we are allowing for the active participation of the hider as opposed to other formulations which require a stationary hider. An n-cell search game, as we call it, and two sequential extensions of the n-cell game are proposed.

We summarize our game formulations and results. In the n-cell game, every play consists of exactly one move. The payoff function is taken to be the probability that P1 detects P2. This payoff is particularly appealing for Antisubmarine Warfare (ASW) applications. With the indicated payoff, a zero-sum assumption corresponds to the rôle of an evader for P2. To illustrate this point, an example of a typical ASW situation is given. We also show how a constrained-game extension can be employed to include additional tactical information.

The n-cell game is extended to a sequential game. In turn, we consider sequential games consisting of a finite number and an infinite number of moves. On each move, the players not only determine an immediate payoff, but they also determine the probability that the game terminates

[1] For an example, see Koopman [9] and Bellman [1].

before the next move. This game may be thought of as a matrix game which is played again with probability determined by the players.

For the finite sequential game, we show how optimal strategies and the value may be computed by dynamic programming. In particular, the solution of the game can be found by solving a recursive sequence of matrix games. The amount of computational effort is usually much less than would be required by solution of the game in normal form.

When an infinite number of moves is allowed, we show how to characterize the resulting sequential game by a special type of non-linear program. If one of the variables in the constraint set of this non-linear program is held fixed, it becomes a linear program. To find the solution of the sequential game, we must adjust this variable in the constraint set to make the optimal value of the objective function equal to zero. We show how to perturb the linear program and thereby approximate the game solution to within desired accuracy.

Many of the search models which appear in the literature assume a stationary hider. Models of this type are given by Koopman [9], Bellman [1], Pollock [15], Dobbie [7], and MacQueen [12]. On the other hand, game formulations which allow the hider to choose strategies are presented by von Neumann [19], Norris [14], and Neuts [13]. Our games are generalizations and extensions of these search games. In particular, one of von Neumann's [19] search games is a special case of the n-cell search game which is presented in section 2.

2. THE N-CELL SEARCH GAME

2.1 Formulation

To formulate the n-cell search game, we assume that the searcher has one detection device and that there is only one hider. Later, we relax these assumptions. The search region of interest is divided into n-cells. A pure strategy for the searcher (P1) is a cell to search (locate his detection device) and a pure strategy for the hider (P2) is a cell in which to hide. A play consists of exactly one simultaneous choice of strategies (move) or, of course, the players may choose their strategies sequentially provided the second choice is made in ignorance of the first.

To define the payoffs, we assume that if P1 looks in cell i and P2 hides in cell j, then P1 detects P2 with probability a_{ij} $(i, j = 1, \ldots, n)$. Let A be the $n \times n$ payoff matrix $A = (a_{ij})$.

To embrace tactical encounters, we have postulated the payoff as a probability of detection. Other search payoffs could be used as well. We have also allowed the probability of detection to be a function of the range between P1 and P2. This feature is not included in most other detection models.

Now, suppose that P1 searches cell i with probability x_i, and suppose that P2 hides in cell j with probability y_j. We require that

$$\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \quad i = 1, \ldots, n \quad ,$$

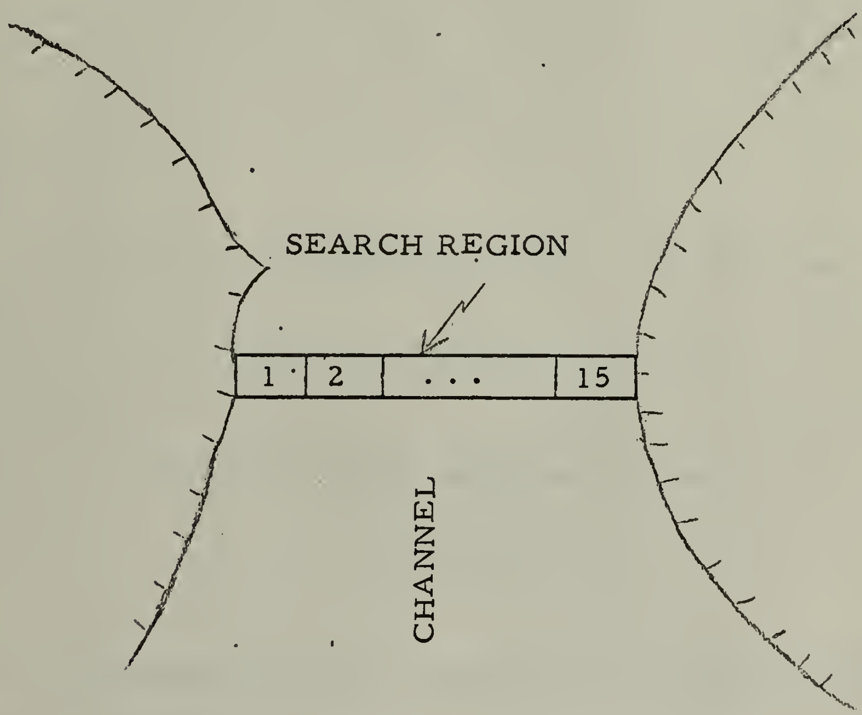
$$\sum_{j=1}^{n} y_j = 1, \quad y_j \geq 0, \quad j = 1, \ldots, n \quad .$$

Let X and Y be the nx1 vectors $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$. Then X and Y are mixed strategies for P1 and P2, respectively. If P1 chooses X and P2 chooses Y then, from elementary probability, P1 detects P2 with probability $X^t A Y$.[1] We assume that P1 chooses X to maximize his detection probability $X^t A Y$. Now consider the case when P2 chooses Y to minimize $X^t A Y$. Then we can interpret the motive of P2 as evasive action since P2 is attempting to minimize the probability that he is detected. This action by P2 also gives rise to a zero-sum game. It is important to note this relationship between evasion and a zero-sum game. We henceforth restrict our discussion to zero-sum games.

[1] X^t denotes the vector X - transpose.

2.2 An Example

To illustrate the n-cell game, consider the following tactical example. Suppose a submarine must pass through a channel to get from its base to its patrol area.[1] The searcher, P1, wishes to locate a detection device in the channel to detect submarines as they pass through. For convenience, we assume that the search region is divided into 15 cells, as shown in Figure 1. We assume that P1 wants to locate his device in one of these cells to maximin the probability of detection.

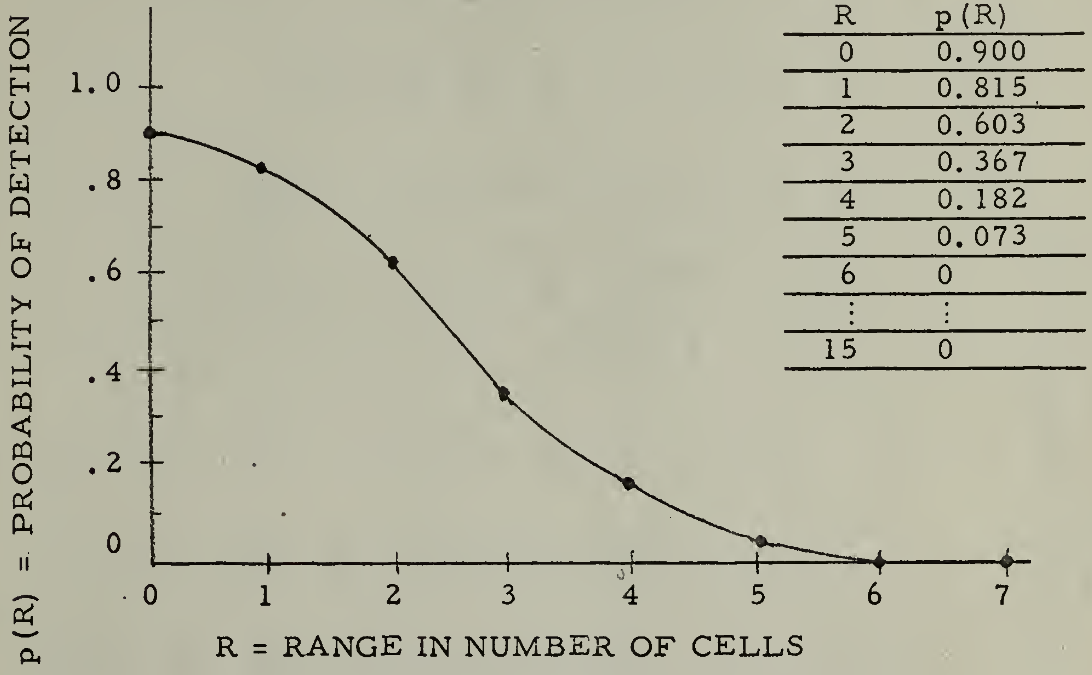


The Search Region and Cells

Figure 1

[1] This type of situation was encountered in the Bay of Biscay during World War II, Sternhell and Thorndike [18].

To obtain the payoff matrix, we need the detection probabilities. These probabilities are found from the probability of detection versus range curve, as shown in Figure 2.



R	p(R)
0	0.900
1	0.815
2	0.603
3	0.367
4	0.182
5	0.073
6	0
⋮	⋮
15	0

Probability of Detection versus Range

Figure 2

To find the payoff matrix, suppose that P1 locates his detection device in cell 5, and P2 hides in cell 8; then the range is three cells, and the probability of detection, from Figure 2, is $a_{52} = 0.367$. The other elements of the payoff matrix A are determined in a similar manner, and the complete matrix is given in Figure 3.

PLAYER 2'S PURE STRATEGIES

PLAYER 1'S PURE STRATEGIES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	.900	.815	.603	.367	.182	.073									
2	.815	.900	.815	.603	.367	.182	.073								
3	.603	.815	.900	.815	.603	.367	.182	.073							
4	.367	.603	.815	.900	.815	.603	.367	.182	.073						
5	.182	.367	.603	.815	.900	.815	.603	.367	.182	.073					
6	.073	.182	.367	.603	.815	.900	.815	.603	.367	.182	.073				
7		.073	.182	.367	.603	.815	.900	.815	.603	.367	.182	.073			
8			.073	.182	.367	.603	.815	.900	.815	.603	.367	.182	.073		
9				.073	.182	.367	.603	.815	.900	.815	.603	.367	.182	.073	
10					.073	.182	.367	.603	.815	.900	.815	.603	.367	.182	.073
11						.073	.182	.367	.603	.815	.900	.815	.603	.367	.182
12							.073	.182	.367	.603	.815	.900	.815	.603	.367
13								.073	.182	.367	.603	.815	.900	.815	.603
14									.073	.182	.367	.603	.815	.900	.815
15										.073	.182	.367	.603	.815	.900

All blank elements are zeros.

Player 1's Payoff Matrix

Figure 3

The value and optimal strategies (solution) of the game can now be computed. The solution was computed by linear programming, and it is tabulated in Figure 4. From this figure, we observe how the boundaries of the search region affect the searching strategy. A substantial amount of the effort is allocated to the end cells. Notice that the only data required for this search model is a probability of detection versus range curve.

j	y_j
1	0.258
5	0.069
6	0.165
10	0.234
15	0.274

all other $y_j = 0$

j	y_j
1	0.274
6	0.234
10	0.165
11	0.069
15	0.258

all other $y_j = 0$

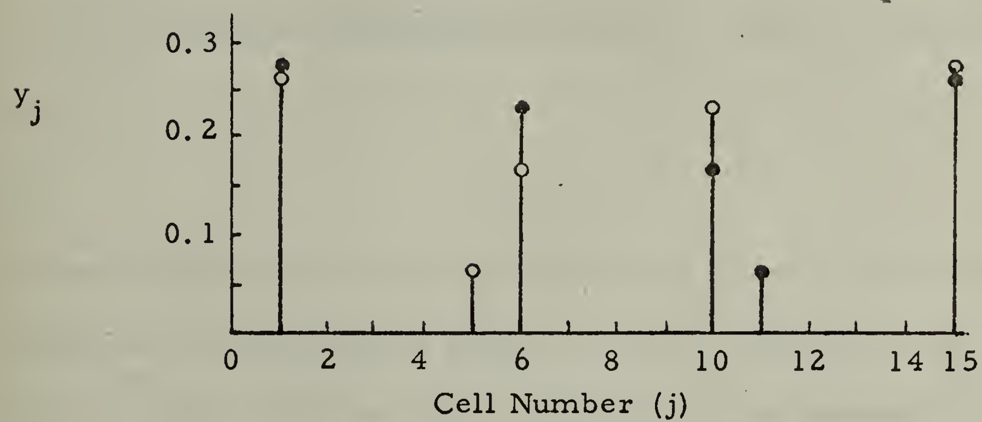
P2's Optimal Basic Strategies

j	x_j
2	0.317
6	0.099
8	0.168
10	0.099
14	0.317

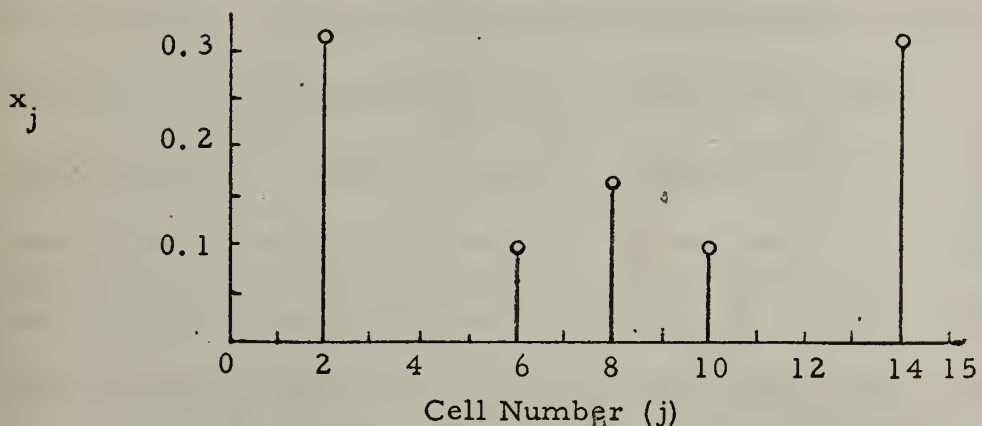
all other $x_j = 0$

P1's Optimal Strategy

4a



P2's Optimal Strategies
4b



P1's Optimal Strategy
4c

OPTIMAL STRATEGIES
Figure 4

2.3 Extensions

Before leaving this subject, we point out some possible extensions. Additional probabilistic information can be included in the n-cell game by considering a constrained game extension. The elegant development of a constrained game by Charnes [2] can then be applied directly. To illustrate the type of constraints which may arise, suppose that the searcher can bound the probability that the hider chooses certain cells, i.e., the searcher determines numbers L_j and U_j $(0 \le L_j \le U_j \le 1)$ such that

$$L_j \le y_j \le U_j \ . \tag{1}$$

These bounds may arise from intelligence or previous contacts. Constraints of the type in equation (1) and, in general, any linear inequalities can be included in a constrained game formulation. The method of Charnes [2] can then be employed.

It is also desirable to relax the assumption that P1 has only one detection device. This can be easily accomplished by redefining P1's pure strategies in the following way. For simplicity, suppose P1 has two detection devices. Then let each pure strategy for P1 be the two-tuple (k, i) where cell k denotes the location of device 1 and cell i is the location of device 2. Now the probability of detection can be calculated for each such pure strategy, and again an ordinary matrix game is obtained. In a similar way, we can also allow P2 to consist of two or more hiders. Next, we formulate a finite sequential version of the n-cell search game.

3. THE FINITE SEQUENTIAL GAME

3.1 Formulation

First, we discuss the elements of the finite sequential game, and then we proceed with the mathematical formulation. A play of the game consists of, at most, a finite number (N) of moves. On each move, when the game has not terminated, the players are faced with a two-person zero-sum game. In our formulation, we shall use the n-cell game as the two-person zero-sum game for each move. When the players move, they each choose a strategy which determines a zero-sum payoff from player 2 to player 1 and a probability that the game terminates before the next move. We wish to find an optimal strategy for each player which minimaxes the expected accumulated payments received by player 1.

The recursive optimization technique which we will propose has also been discussed by other authors. Kuhn [10] (1953) gave his theorem on games of perfect recall which paved the way for further work. Shapley [17] (1953) was the first to point out the recursive character of a generalization of this game, although he did not deal with the finite case. Later contributions were made by Bellman [1] (1957), Everett [8] (1957), Zachrisson [22] (1964), and Denardo [6] (1965).

The payoffs and continuation probabilities are now specified. Suppose that P1 searches cell i and P2 hides in cell j on move r. Then the payoff

from P2 to P1 is

$$a_{ij}(r) \qquad i, j = 1, \dots, n$$
$$r = 1, \dots, N \ .$$

Also, when P1 searches cell i and P2 hides in cell j on move r, the game continues until move r + 1 with probability

$$p_{ij}(r) \qquad i, j = 1, \dots, n$$
$$r = 1, \dots, N - 1 \ .$$

We let A_r be the nxn matrix $A_r = (a_{ij}(r))$ and P_r the nxn matrix $P_r = (p_{ij}(r))$. Hence, A_r is P1's payoff matrix for move r and P_r is the matrix of continuation probabilities for move r. We assume that the game is zero sum and that P1 is the maximizing player.

Next, we consider strategies for the players. We have assumed that the continuation probability and payoff depend only on the choices available for a particular move. It follows that the game is one of perfect recall as defined by Kuhn [10]. Kuhn's theorem for a game of perfect recall asserts that a "behavior strategy" is optimal. For this particular game, a behavior strategy takes the following form: let X_r and Y_r be mixed strategies over the alternatives available on move r for P1 and P2, respectively. Let $X = (X_1, \dots, X_N)$ be an N-tuple of the above mixed strategies for P1. Then X is a behavior strategy for P1. Similarly, we define a behavior strategy, $Y = (Y_1, \dots, Y_N)$, for P2. Now the following sets of strategies are introduced:

$$\mathfrak{X}_r = \{X_r\}, \ \mathfrak{Y}_r = \{Y_r\}, \ \mathfrak{X} = \{X\}, \ \mathfrak{Y} = \{Y\} \ .$$

From Kuhn's theorem of perfect recall, the sets $\mathfrak{X}$ and $\mathfrak{Y}$ contain optimal game strategies for P1 and P2, respectively; and we will, therefore, limit our search for optimal strategies to these sets.

The total expected payoff for P1 will be expressed in terms of fixed strategies $X \in \mathfrak{X}$, $Y \in \mathfrak{Y}$ and the given information. If P1 chooses the strategy $X_r \in \mathfrak{X}_r$ for move r and P2 chooses $Y_r \in \mathfrak{Y}_r$, then the payoff to P1 for move r is

$$X_r^t A_r Y_r \qquad r = 1, \ldots, N$$

and the game continues until move r + 1 with probability

$$X_r^t P_r Y_r \qquad r = 1, \ldots, N-1 .$$

Now the product of the probability that the game continues until move r and the payoff for move r is

$$X_r^t A_r Y_r \prod_{h=1}^{r-1} X_h^t P_h Y_h \qquad r = 2, 3, \ldots, N .$$

The expected accumulated payoff for N moves, $v_1(X, Y)$, is the sum of the above terms

$$(2) \qquad v_1(X, Y) = X_1^t A_1 Y_1 + \sum_{r=2}^{N} X_r^t A_r Y_r \prod_{h=1}^{r-1} X_h^t P_h Y_h .$$

Since the game has a finite number of moves and a finite number of strategies, it must have a value and optimal strategies.[1] Recall that the sets $\mathfrak{X}$ and $\mathfrak{Y}$ contain optimal strategies. Therefore, the function $v_1(X, Y)$ has at least one saddle point over the sets $\mathfrak{X}$ and $\mathfrak{Y}$.

[1] von Neumann and Morgenstern [21].

3.2 Recursive Solution

We show how to compute the minimax of equation (2) by a recursive technique. Let $\overline{X}_r$ and $\overline{Y}_r$ denote the sequences of mixed strategies

$$\begin{aligned} \overline{X}_r &= (X_r, X_{r+1}, \ldots, X_N) \\ & \qquad\qquad\qquad\qquad r = 1, \ldots, N \; . \\ \overline{Y}_r &= (Y_r, Y_{r+1}, \ldots, Y_N) \end{aligned}$$

Of course, we have $\overline{X}_1 \equiv X$, and $\overline{Y}_1 \equiv Y$. We rewrite equation (2) and also define the scalar functions $v_r(\overline{X}_r, \overline{Y}_r)$ by

$$(3) \quad v_r(\overline{X}_r, \overline{Y}_r) = X_r^t A_r Y_r + (X_r^t P_r Y_r)\, v_{r+1}(\overline{X}_{r+1}, \overline{Y}_{r+1}) \qquad r = 1, \ldots, N$$

$$v_{N+1} \equiv 0$$

Now $v_r(\overline{X}_r, \overline{Y}_r)$ may be interpreted as the expected accumulated payments received by P1 on the last N - r + 1 moves of the game.

Equation (3) leads us to believe that the recursive optimization technique of dynamic programming can be employed. We will establish this fact by theorem 1. We define $\hat{v}_1$, $\hat{X}_r$, $\hat{Y}_r$ by the following equations

$$(4) \quad \hat{v}_r = \underset{X_r \in \mathfrak{X}_r}{\mathrm{Max}} \; \underset{Y_r \in \mathfrak{Y}_r}{\mathrm{Min}} \left[X_r^t A_r Y_r + (X_r^t P_r Y_r)\, \hat{v}_{r+1} \right] \qquad r = 1, \ldots, N$$

$$\hat{v}_{N+1} \equiv 0$$

$$\equiv \hat{X}_r^t A_r \hat{Y}_r + (\hat{X}_r^t P_r \hat{Y}_r)\, \hat{v}_{r+1} \; .$$

The minimax theorem of von Neumann [21] establishes the existence of $\hat{X}_r$, $\hat{Y}_r$, $\hat{v}_r$ as defined by equation (4). The following theorem then relates the solutions of equation (4) to the solutions of the sequential game.

Theorem 1: $\hat{v}_1$ is the value of the sequential game, and $\hat{X} = (\hat{X}_1, \ldots, \hat{X}_N)$, $\hat{Y} = (\hat{Y}_1, \ldots, \hat{Y}_N)$ are optimal strategies for P1 and P2, respectively.

Proof: Since $v_1(X, Y)$ from equation (3) is the expected payoff function for the sequential game, a necessary and sufficient condition for $\hat{v}_1$ to be the value of the game and $\hat{X}$, $\hat{Y}$ optimal strategies is

$$v_1(X, \hat{Y}) \le \hat{v}_1 \le v_1(\hat{X}, Y) \quad \text{all } X \in \mathfrak{X} \text{ and } Y \in \mathfrak{Y} .$$

We shall show that the above condition is satisfied by $\hat{v}_1$, $\hat{X}$, $\hat{Y}$ as defined by (4). From (4), we have

$$(5) \quad X_r^t A \hat{Y}_r + (X_r^t P_r \hat{Y}_r)\hat{v}_{r+1} \le \hat{v}_r \le \hat{X}_r^t A Y_r + (\hat{X}_r^t P_r Y_r)\hat{v}_{r+1} ,$$

$$\text{all } X_r \in \mathfrak{X}_r, \quad Y_r \in \mathfrak{Y}_r .$$

We begin an inductive argument

$$v_N(\hat{X}_N, Y_N) = \hat{X}_N^t A_N Y_N \ge \hat{v}_N \quad \text{all } Y_N$$

assume

$$v_{r+1}(\hat{\overline{X}}_{r+1}, \overline{Y}_{r+1}) \ge \hat{v}_{r+1} \quad \text{for some } r \text{ and all } \overline{Y}_r .$$

By definition,

$$v_r(\hat{\overline{X}}_r, \overline{Y}_r) = \hat{X}_r^t A_r Y_r + (\hat{X}_r^t P_r Y_r)\, v_{r+1}(\hat{\overline{X}}_{r+1}, \overline{Y}_{r+1}) .$$

By the inductive assumption and $\hat{X}_r^t P_r Y_r \ge 0$,

$$v_r(\hat{\overline{X}}_r, \overline{Y}_r) \ge \hat{X}_r^t A_r Y_r + (\hat{X}_r^t P_r Y_r)\hat{v}_{r+1} \quad \text{all } \overline{Y}_r .$$

From the preceding equation and equation (5)

$$v_r(\hat{\overline{X}}_r, \overline{Y}_r) \geq \hat{v}_r \quad \text{all } \overline{Y}_r .$$

Hence, by induction on r [1].

$$v_1(\hat{X}, Y) \geq \hat{v}_1 \quad \text{all } Y \in \mathbf{Y} .$$

Similarly, we may establish

$$v_1(X, \hat{Y}) \leq \hat{v}_1 \quad \text{all } X \in \mathbf{X} ;$$

therefore,

$$v_1(X, \hat{Y}) \leq \hat{v}_1 \leq v_1(\hat{X}, Y) \quad \text{all } X \in \mathbf{X}, \; Y \in \mathbf{Y}$$

and the theorem is true.

In order to find the value and optimal strategies, we can solve equation (4) recursively. But, each iteration of equation (4) requires the solution of an ordinary matrix game. Now the solutions (value and optimal strategies) of a matrix game can be found by solving a linear programming formulation of the game.[2] Hence, we can find the solutions of the sequential game by optimizing a sequence of N linear programs. Of course, the amount of computational effort required by this method will usually be substantially less than the amount required to solve the sequential game in direct normal form.

[1] Notice that by definition of $\overline{X}_1$ we have $\hat{\overline{X}}_1 \equiv \hat{X} \equiv (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_N)$.

[2] See Charnes and Cooper [3].

4. THE INFINITE SEQUENTIAL GAME

4.1 Formulation

In this section, we allow an infinite number of moves in the sequential game. Before giving an analytic formulation, we discuss some of the features of the game. In the infinite sequential game, we do not assume a maximum number of moves. The continuation probabilities alone control the termination of the game. However, we assume that the same payoff matrix and the same continuation probability matrix are specified for all moves. We further assume that the probability of continuing until the next move is strictly less than one for all pairs of strategies. This assumption guarantees boundedness of the expected accumulated payments received by P1 ; and it guarantees that the game terminates with probability one, although the number of moves may not be bounded. Now we turn to a formal definition of the game under consideration.

As in the previous games, we assume that a search region is specified and that it is divided into n cells. If P1 chooses cell i $(i = 1, \ldots, n)$ and P2 chooses cell j $(j = 1, \ldots, n)$ on move r $(r = 1, 2, \ldots)$, then P1 receives from P2 the payoff

$$a_{ij}$$

and the game continues until move r + 1 with probability

(6) $$0 \leq p_{ij} < 1 \ .$$

Let P be the n x n matrix $P = (p_{ij})$ and A the n x n matrix $A = (a_{ij})$. A is the payoff matrix, and P is the matrix of continuation probabilities for every move. We further assume that the game is zero sum and that P1 is the maximizing player.

The game which we have defined above is one of "perfect recall"; and by Kuhn's [10] theorem, a "behavior strategy" is optimal. If a player uses a behavior strategy, he plays the same mixed strategy over the alternatives in an information set each time the information set is reached, regardless of the past history of the game. Since the matrices A and P apply to every move, the game has only one information set. Therefore, a behavior strategy is simply a mixed strategy which is used for every move of the game. We restrict our attention to these strategies.

Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be behavior strategies (mixed strategies over the alternatives) for P1 and P2, respectively. For example, P1 chooses alternative i with probability x_i on every move. The expected accumulated payment received by P1, $v(X, Y)$, when P1 chooses X and P2 chooses Y, is simply the sum over all r of the probability that the game lasts until move r times the payment to P1 for move r,

$$v(X, Y) = \sum_{r=0}^{\infty} (X^t P Y)^r X^t A Y . \tag{7}$$

The above sum converges, since (6) implies $0 \leq X^t P Y < 1$ for all strategies X and Y. For convenience, we define the matrix $Q = (q_{ij})$ with $q_{ij} = 1 - p_{ij}$ all i, j. Then Q is the matrix of positive termination

probabilities. Equation (7) may be written as

$$v(X, Y) = \frac{X^t A Y}{1 - X^t P Y} = \frac{X^t A Y}{X^t Q Y} \quad . \tag{8}$$

von Neumann [20] first established the existence of a unique value v and optimal strategies $\hat{X}$ and $\hat{Y}$ for the form in (8), i.e., there exists a unique real number v and strategies $\hat{X}$, $\hat{Y}$ such that

$$\frac{X^t A \hat{Y}}{X^t Q \hat{Y}} \leq v \leq \frac{\hat{X}^t A Y}{\hat{X}^t Q Y} \quad \text{all strategies } X, Y \ . \tag{9}$$

An elementary proof of this fact was subsequently given by Loomis [11], and this result is a special case of Shapley's [17] more general "stochastic game". Neuts [13] formulated and solved a special case of the infinite sequential game. His P matrix was a diagonal matrix and his A matrix also had a special form.

4.2 Solution by Perturbation

There are no known methods for computing a solution to equation (9). In this section, we develop a computational method to approximate a solution to any desired degree to accuracy. The method is based on a linear programming formulation of a matrix game with an unknown parameter in the constraints. We show that this parameter is equal to the value of the game if and only if the optimal objective function of the linear program is zero. The remainder of our discussion is then devoted to a method for approximating the required value of the parameter.

To begin, we establish a lemma which relates the solution of the infinite sequential game to the solution of an ordinary two-person zero-sum game.

<u>Lemma:</u> A necessary and sufficient condition for v to be the value of the infinite sequential game and $\hat{X}$, $\hat{Y}$ optimal strategies is that the two-person zero-sum game with payoff matrix $A - vQ$ has value zero and optimal strategies $\hat{X}$, $\hat{Y}$.

<u>Proof</u>: For the matrix game $A - vQ$ to have value zero and optimal strategies $\hat{X}$, $\hat{Y}$, it is necessary and sufficient that

$$(10) \qquad X^t (A - vQ)\, \hat{Y} \le 0 \le \hat{X}^t (A - vQ)\, Y \quad \text{all strategies } X, Y .$$

But, $X^t Q Y > 0$ for all strategies, X, Y. Hence, v, X, Y satisfy (10) if and only if

$$(10a) \qquad \frac{X^t A \hat{Y}}{X^t Q \hat{Y}} \le v \le \frac{\hat{X}^t A Y}{\hat{X}^t Q Y} \quad \text{all strategies } X, Y .$$

Equation (10a) is a necessary and sufficient condition for v to be the value of the infinite sequential game and $\hat{X}$, $\hat{Y}$ optimal strategies. Hence, the lemma is true.

This lemma immediately suggests a method for computing v. The general procedure is to choose a number s and compute the value of the game $A - sQ$. If the value of $A - sQ$ is zero, then $s = v$ and we are finished. If the value of $A - sQ$ is not zero, then we want to choose a new value of s, say s_1, such that the value of $A - s_1 Q$ is "closer" to zero than the value of $A - sQ$. We begin by formulating the matrix game $A - sQ$ as a linear program.

Consider the linear program

$$\text{(11)} \qquad \begin{aligned} &\text{Max } u_s \\ &u_s e^t - X^t (A - sQ) \le 0 \\ &X^t e = 1 \\ &X \ge 0 \end{aligned}$$

where e is the n x l vector of all "ones", X is an n x l vector, s is a fixed scalar, and u_s is a scalar variable. Let $\hat{u}_s$, $\hat{X}$ be an optimal solution to (11). Then from Charnes [2], $\hat{X}$ is an optimal strategy for P1 and $\hat{u}_s$ is the value of the game $A - sQ$, (s fixed). Of course, an optimal strategy $\hat{Y}$ for P2 is part of an optimal solution to the dual of (11), and $\hat{Y}$ is available when (11) is solved by the simplex method.

Next, we examine the variation in $\hat{u}_s$ which results from a change

in s. This will allow us to perturb s in such a way that we move $\hat{u}_s$ closer to zero. We consider a perturbation from s to $s + \xi$ in problem (11), and we want to relate $\hat{u}_s$ to $\hat{u}_{s+\xi}$. We add and subtract the vector $\xi X^t Q$ from the constraints of (11) and obtain the following equivalent linear program

$$\text{Max } u_s$$

$$(12) \qquad u_s e^t - X^t (A - (s + \xi) Q) - \xi X^t Q \le 0$$

$$X^t e = 1$$

$$X \ge 0 .$$

We seek to obtain a linear programming formulation of the game $A - (s + \xi) Q$ from (12). Hence, we let $\overline{\overline{q}} = \max_{i,j} q_{ij}$, $\overline{q} = \min_{i,j} q_{ij}$ and then for $\xi > 0$

$$(13) \qquad \xi \overline{q} e^t \le \xi X^t Q \le \xi \overline{\overline{q}} e^t \quad \text{all strategies } X .$$

Now consider the following linear program

$$\text{Max } u'$$

$$(14) \qquad u' e^t - X^t (A - (s + \xi) Q) \le \xi \overline{\overline{q}} e^t$$

$$X^t e = 1$$

$$X \ge 0 .$$

Problem (14) is "less constrained" than (12). Therefore, the respective optimal solutions must satisfy ("hats" on the variables denote optimal values)

$$(15) \qquad \hat{u}' \ge \hat{u}_s .$$

Notice that the right-hand side of the constraints in (14) is a constant vector. We bring this vector over to the left-hand side of the constraints and make the change of variable

(16) $$u = u' - \xi \overline{\overline{q}}$$

to obtain the program

$$\text{Max } (u + \xi \overline{\overline{q}})$$

(17) $$u\, e^t - X^t (A - (s + \xi) Q) \leq 0$$

$$X^t e = 1$$

$$X \geq 0 \ .$$

But, (17) is the desired linear programming formulation of the game $A - (s + \xi) Q$ except for the additive constant $+ \xi q$ in the objective function. Hence,

$$\hat{u} = \hat{u}_{s + \xi}$$

and

$$\hat{u}' = \hat{u}_{s + \xi} + \xi \overline{\overline{q}} \ .$$

From (15) and the above equation

$$\hat{u}_{s + \xi} + \xi \overline{\overline{q}} \geq \hat{u}_s \ .$$

By using the left-hand side of (13), we get by a similar argument

$$\hat{u}_{s + \xi} + \xi \overline{q} \leq \hat{u}_s \ .$$

Thus, for $\xi > 0$

(18) $$\hat{u}_s - \xi \overline{\overline{q}} \leq \hat{u}_{s + \xi} \leq \hat{u}_s - \xi \overline{q}, \quad \xi > 0$$

and for $\xi < 0$ we can derive the relationship

$$(19) \qquad \hat{u}_s - \xi\,\overline{q} \le \hat{u}_{s+\xi} \le \hat{u}_s - \xi\,\overline{\overline{q}}, \quad \xi < 0 .$$

Equations (18) and (19) give the desired relationships. We can choose a starting value of s and then subsequently perturb $\hat{u}_s$ toward zero.

To assist in choosing a starting value of s, we propose the following method. Two numbers, m and M ($m \le M$), are determined such that $\hat{u}_m \ge 0$ and $\hat{u}_M \le 0$. Then since $\hat{u}_s$ is a continuous function of s,[1] $\hat{u}_s = 0$ for some s in the range $m \le s \le M$. Furthermore, this value of s is unique. We would then choose the initial value of s to satisfy $m \le s \le M$. Suppose we take m and M to be

$$(20) \qquad m = \min_{i,j} \frac{a_{ij}}{q_{ij}}, \quad M = \max_{i,j} \frac{a_{ij}}{q_{ij}},$$

then

$$m\,q_{ij} \le a_{ij}, \quad M\,q_{ij} \ge a_{ij} \quad \text{all } i, j .$$

From the constraints of (11), we see that

$$\hat{u}_s = \min_j \sum_{i=1}^{n} \hat{x}_i (a_{ij} - s\,q_{ij}) ;$$

thus,

$$\hat{u}_m \ge 0, \quad \hat{u}_M \le 0 .$$

With certain restrictions on the elements a_{ij}, we can derive tighter bounds than m and M; but, the bounds given here are adequate for most applications.

[1] This fact is clear from the foregoing derivation.

4.3 Iterative Method

There are various methods which can be used in connection with equations (18) and (19) to move from one value of s to another. We propose one method here which will automatically change s and drive $\hat{u}_s$ to zero. We find two numbers, b and ξ, for fixed s which will give

$$-b \le \hat{u}_{s+\xi} \le b \quad . \tag{21}$$

By reference to (18) and (19), we find that

$$\xi = \frac{2\hat{u}_s}{\bar{\bar{q}}+\bar{q}} \quad , \quad b = \frac{|\hat{u}_s|\,(\bar{\bar{q}}-\bar{q})}{\bar{\bar{q}}+\bar{q}} \quad . \tag{22}$$

The following method for approximating a solution to equation (9) can now be started. For convenience, let $\hat{u}_i$ be the optimal solution to (11) when s_i is the value of s in the constraint set.

<u>Approximation Method</u>

1. Choose a starting value s_0 and calculate u_0 from (11).
2. Given s_i and $\hat{u}_i$, let

$$s_{i+1} = s_i + \frac{2\hat{u}_i}{\bar{\bar{q}}+\bar{q}}$$

and calculate $\hat{u}_{i+1}$ from (11).

Let $U = (\hat{u}_1, \hat{u}_2, \ldots)$ be the sequence generated by the above method. We now show that this sequence does indeed converge absolutely to zero.

By step 2 and equations (21) and (22), we have

(23) $$|\hat{u}_{i+1}| \le |\hat{u}_i| \; \frac{\bar{\bar{q}} - \bar{q}}{\bar{q} + \bar{\bar{q}}} \; .$$

Let

$$a = \frac{\bar{\bar{q}} - \bar{q}}{\bar{q} + \bar{\bar{q}}} \; ;$$

notice that $0 \le a < 1$, and we also have

$$|\hat{u}_n| \le a^n \; |\hat{u}_0| \; .$$

Since $a^n \longrightarrow 0$, we have $|\hat{u}_n| \longrightarrow 0$. Because of this convergence, we can approximate the value of the game to any desired accuracy by the above method.

We point out a few interesting features of the proposed method. From step 2, the "driving force" which moves us from s_i to s_{i+1} is directly proportional to the magnitude of $\hat{u}_i$. This feature serves to drive $\hat{u}_i$ rapidly toward zero. We also observe that when $\bar{\bar{q}} = \bar{q}$, we will have $\hat{u}_1 = 0$. This is a nice feature since $\bar{\bar{q}} = \bar{q}$ implies that all elements of the Q matrix are equal and, therefore, the game reduces to an ordinary matrix game (see equation (8)).

An alternative to the method which we have proposed here is to use the contraction property of the operator T which is defined as

$$T\alpha = \mathrm{Val}\,[A - \alpha P]$$

where α is a scalar and $\mathrm{Val}[A - \alpha P]$ denotes the value of the matrix game $A - \alpha P$. Shapley [17] has shown in a more general setting that

$T^n\alpha$ approaches the value of the infinite stochastic game for arbitrary α, and Charnes and Schroeder [5] have shown how to obtain a desired approximation to the value or optimal strategies by using linear programming methods.

4.4 Tactical Payoffs and a Special Case

For tactical purposes, two particular payoffs are appealing. As in the n-cell game, the payoff for each move may be the probability that P1 detects P2 in that move. Then, in both sequential games, the expected accumulated payment received by P1 will be the probability that P1 detects P2 in the game. The other payoff of interest is obtained by taking all $a_{ij}(r) = 1$. Then P1 always receives a payoff of one unit for each move, and the expected accumulated payment received by P1 is simply the expected number of moves. When each move takes the same length of time, we may, of course, also interpret this payoff as the expected duration of the game. In one of the games in Charnes and Schroeder [5], we show how to incorporate both of these payoffs in the same formulation. In particular, P1 attempts to maximize the probability of detection while constraining the expected number of moves to be no more than a specified number.

Finally, we show that the infinite sequential game reduces to an ordinary matrix game when P1 attempts to minimize the expected number of moves. From the above discussion, we take all $a_{ij} = 1$ for the expected accumulated payment to be the expected number of moves. Then equation (8) becomes

$$v(X, Y) = \frac{1}{X^t QY} .$$

For search problems, P1 wants to choose a mixed strategy X to

minimize the expected number of moves. Therefore, we seek to solve the equation

$$(24) \qquad v = \min_X \max_Y \frac{1}{X^t Q Y} \equiv \frac{1}{\hat{X}^t Q \hat{Y}} .$$

Clearly, v, $\hat{X}$, and $\hat{Y}$ satisfy (24) if they satisfy

$$\frac{1}{v} = \max_X \min_Y \; X^t Q Y \equiv \hat{X}^t Q \hat{Y} .$$

Hence, P1 can minimax the expected number of moves by maximining the probability that the game terminates in one move. This is a noteworthy feature of the infinite sequential game. Of course, in this case our perturbation technique is not required since we can solve the game directly by means of a single linear program.

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We develop some two-person zero-sum formulations of search and evasion problems. By employing a game theoretic approach, we allow the hider, as well as the searcher, to choose a strategy. This is in contrast to most search models which assume a stationary or passive hider. Both non-sequential and sequential search games are investigated. Some interesting aspects of the non-sequential game and an example of an antisubmarine search problem are given. The sequential games consist of a sequence of moves. When the players move, they not only determine a payoff but also the probability that the game terminates before the next move. When at most a finite number of moves is allowed, we prove that a solution may be found by solving a recursive sequence of matrix games. When the number of moves is not bounded, the game is characterized by a special type of non-linear program. The solution to this program can be approximated by successive perturbations of a related linear program. Finally, we obtain the result that a pair of strategies minimaxes the expected duration of the game if and only if these strategies also maximin the probability of termination in one step.

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